RELATIVE SEIBERG-WITTEN AND OZSVATH-SZABO 4-DIMENSIONAL INVARIANTS WITH RESPECT TO EMBEDDED SURFACES

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§1. INTRODUCTION

1.1. The subject. In this paper we define and study the relative Seiberg-Witten (SW) invariant and an analogous relative Ozsváth-Szabó (OS) invariant of pairs (X, Σ), where Σ is a surface of genus $g > 1$ embedded in a 4-manifold X. This kind of invariant in case of $g = 1$ was constructed by Taubes [T5]. It was observed in [T5] and then ingeniously used in [FS1] and [FS2] that the relative SW invariant of Taubes for (X, Σ) is practically reduced to the absolute SW invariant of the fiber sum of X with $E(1)$.

This property suggested an elementary definition of a similar invariant for (X, Σ), $g > 1$, as the pull-back of the absolute invariant of a certain fiber sum $X\#_\Sigma W$. The same approach works for the OS invariants. Our goal is to show that the choice of W is not essential provided it admits a relatively minimal Lefschetz fibration $W \to S^2$ with a fiber Σ and has $H_1(W) = 0$. We obtain also a package of properties for these relative invariants which is analogous to the package of properties of the absolute invariants.

Briefly speaking, if a basic $\mathrm{Spin}^{\mathbb{C}}$ structure, $\mathfrak{s}$, in SW (or in OS) theory is extremal with respect to the adjunction inequality for Σ, that is belongs to the image, $\mathrm{Spin}^{\mathbb{C}}(X, \mathfrak{s}_\Sigma) = \{x \in \mathrm{Spin}^{\mathbb{C}}(X) | c_1(\mathfrak{s}) = \chi(\Sigma) + \Sigma^2\}$, of the forgetful map $abs_{X,\Sigma}\colon \mathrm{Spin}^{\mathbb{C}}(X, \Sigma) \to \mathrm{Spin}^{\mathbb{C}}(X)$, then $abs^{-1}_{X,\Sigma}(\mathfrak{s})$ contains several relative basic structures, $\mathfrak{r}_i \in \mathrm{Spin}^{\mathbb{C}}(X, \Sigma)$, and the SW (as well as OS) invariant of $\mathfrak{s}$ splits into a sum of the relative invariant of $\mathfrak{r}_i$. The relative invariants carry a package of properties analogous to those of the absolute invariants.

The manifolds that we consider in what follows are smooth, oriented, connected, and closed, unless we state otherwise. We suppose also that the surface $\Sigma \subset X$ has genus $g > 1$ and either essential with self-intersection $\Sigma^2 = 0$, or has $\Sigma^2 > 0$ (in the latter case we blow it up to obtain $\Sigma^2 = 0$). This implies in particular that $b_2^+(X) \geqslant 1$.

1.2. The Seiberg-Witten and the Ozsváth-Szabó invariants. In its simplest version, the SW invariant of a 4-manifold X is a function on the set of $\mathrm{Spin}^{\mathbb{C}}$ structures, $SW_X\colon \mathrm{Spin}^{\mathbb{C}}(X) \to \mathbb{Z}$, which takes non-zero values only at a finite set of $\mathfrak{s} \in \mathrm{Spin}^{\mathbb{C}}(X)$, called *the basic structures*, whose degree $d(\mathfrak{s}) = \frac{1}{4}(c_1^2(\mathfrak{s}) - (2\chi(X) + 3\sigma(X))$ is zero. The corresponding Ozsváth-Szabó invariant (extracted as the reduced form of $\Phi_{X,\mathfrak{s}}$ in [OS4],§4) is an analogous function $OS_X\colon \mathrm{Spin}^{\mathbb{C}}(X) \to \mathbb{Z}/\pm$ well-defined up to sign. The sign of $SW_X(\mathfrak{s})$ depends on the choice of an orientation of $H^1(X;\mathbb{R}) \oplus H^2_+(X;\mathbb{R})$ called the *homology orientation*, which is supposed to be fixed. In a bit special case of $b_2^+(X) = 1$ the invariants SW_X, OS_X depend on some additional data that must be fixed: SW_X depends on an orientation of the line $H^2_+(X)$ (see [KM], [MST], or [T1]), while OS_X, according to [OS4, Prop. 2.6], depend on the choice of an isotropic line in $H_2(X)$.

One can consider also a more refined and sophisticated version of SW_X and OS_X, which may take non-zero values for $\mathfrak{s} \in \mathrm{Spin}^{\mathbb{C}}(X)$ of degree $d(\mathfrak{s}) > 0$. These values are homomorphisms $SW_{X,\mathfrak{s}} \colon \mathbb{A}_X \to \mathbb{Z}$, $OS_{X,\mathfrak{s}} \colon \mathbb{A}_X \to \mathbb{Z}/\pm$, supported in the set of homogeneous elements of degree $d(\mathfrak{s})$ from the graded ring $\mathbb{A}_X = \Lambda(H_1(X)/\operatorname{Tors}) \otimes \mathbb{Z}[U]$, where Λ stands for the exterior algebra and the grading is defined on the generators, so that U has degree 2 and $\alpha \in H^1(X) \smallsetminus \{0\}$ have degree 1. The dual consideration, which is more convenient for us, interprets the refined version of SW_X as a map $\mathrm{Spin}^{\mathbb{C}}(X) \to \mathbb{A}_X^* = \Lambda(H^1(X)) \otimes \mathbb{Z}[U] \cong \Lambda(H^1(X; \mathbb{Z}[U]))$, such that $SW_X(\mathfrak{s})$ is homogeneous of degree $d(\mathfrak{s})$. Reducing the values of OS_X modulo 2, we obtain a similar map $\mathrm{Spin}^{\mathbb{C}}(X) \to \mathbb{A}_X^* \otimes \mathbb{Z}/2$.

We will use notation $S_X \colon \mathrm{Spin}^{\mathbb{C}}(X) \to R_X$ for any of the invariants SW_X or OS_X, either in the refined or in the reduced form. R_X here is a ring $\mathbb{A}_X^*$ or $\mathbb{A}_X^* \otimes \mathbb{Z}/2$ in case of the refined forms of SW_X or OS_X. In case of the reduced forms, R_X is just $\mathbb{Z}$ or $\mathbb{Z}/2$. The relative version, $R_{X,\Sigma}$, of this ring is obtained by replacing $H^1(X)$ by $H^1(X, \Sigma)$ in the definition (so that $R_{X,\Sigma} = R_X$ for the reduced forms of SW and OS invariants).

The formal sum

$$\underline{S}_X = \sum_{\mathfrak{s} \in \mathrm{Spin}^{\mathbb{C}}(X)} S_X(\mathfrak{s}) \,\text{.}\, \mathfrak{s}$$

can be considered as an element of the principal (affine) module $R_X[\mathrm{Spin}^{\mathbb{C}}(X)]$ over the group ring $R_X[H^2(X)]$.

Remark. All the constructions and the results obtained below for SW and OS invariants concern in fact any function S_X satisfying a few basic properties of SW and OS invariants, namely A1–A5 formulated in §3.

1.3. Definition of the relative invariant $S_{X,\Sigma}$. Let $\mathrm{Spin}^{\mathbb{C}}(X, \Sigma)$ denote the set of relative $\mathrm{Spin}^{\mathbb{C}}$ structures and $abs_{X,\Sigma} \colon \mathrm{Spin}^{\mathbb{C}}(X, \Sigma) \to \mathrm{Spin}^{\mathbb{C}}(X)$ the forgetful map. Gluing of relative $\mathrm{Spin}^{\mathbb{C}}$ structures in a fiber sum $X_+ \#_\Sigma X_-$ (see 2.11) yields

$$\vee \colon \mathrm{Spin}^{\mathbb{C}}(X_+, \Sigma) \times \mathrm{Spin}^{\mathbb{C}}(X_-, \Sigma) \to \mathrm{Spin}^{\mathbb{C}}(X_+ \#_\Sigma X_-, \Sigma), \quad (\mathfrak{r}_+, \mathfrak{r}_-) \mapsto \mathfrak{r}_+ \vee \mathfrak{r}_-$$

whose composition with the forgetful map gives

$$\#_\Sigma \colon \mathrm{Spin}^{\mathbb{C}}(X_+, \Sigma) \times \mathrm{Spin}^{\mathbb{C}}(X_-, \Sigma) \to \mathrm{Spin}^{\mathbb{C}}(X_+ \#_\Sigma X_-), \quad (\mathfrak{r}_+, \mathfrak{r}_-) \mapsto \mathfrak{r}_+ \#_\Sigma \mathfrak{r}_-.$$

Choose any relatively minimal Lefschetz fibration $W \to S^2$ with a fiber Σ and $b_1(W) = 0$ and denote by $\mathfrak{r}_{W,\Sigma} \in \mathrm{Spin}^{\mathbb{C}}(W, \Sigma)$ its canonical relative $\mathrm{Spin}^{\mathbb{C}}$ structure of the Lefschetz fibration introduced in 2.7. If $\Sigma^2 = 0$, then we define for any $\mathfrak{r} \in \mathrm{Spin}^{\mathbb{C}}(X, \Sigma)$

$$S_{X,\Sigma}(\mathfrak{r}) = S_{X \#_\Sigma W}(\mathfrak{r} \#_\Sigma \mathfrak{r}_{W,\Sigma})$$

If $\Sigma^2 > 0$, then we blow up X at points of Σ to obtain $\hat{X}$, with $\Sigma^2 = 0$, and let

$$S_{X,\Sigma}(\mathfrak{r}) = S_{\hat{X},\Sigma}(\hat{\mathfrak{r}})$$

where $\hat{\mathfrak{r}}$ is the image of $\mathfrak{r}$ under the natural map $\mathrm{Spin}^{\mathbb{C}}(X, \Sigma) \to \mathrm{Spin}^{\mathbb{C}}(\hat{X}, \Sigma)$ (see 2.10).

We will let

$$\underline{S}_{X,\Sigma} = \sum_{\mathfrak{r} \in \mathrm{Spin}^{\mathbb{C}}(X,\Sigma)} S_{X,\Sigma}(\mathfrak{r}) \,\text{.}\, \mathfrak{r} \in R_{X,\Sigma}[\mathrm{Spin}^{\mathbb{C}}(X, \Sigma)]$$

Remarks.

(1) Note that $b_2^+(X\#_\Sigma W)>1$, so $S_{X\#_\Sigma W}$ is well defined.
(2) The differential type of $X\#_\Sigma W$ may depend in principle on the framing of Σ in X and W, so proving that $S_{X,\Sigma}$ is independent of W implies also independence of the framings.
(3) In the case $S_X = SW_X$ we should take care of the homology orientation for $X\#_\Sigma W$. It is determined by the given homology orientation of X and the canonical symplectic homology orientation of W (defined in [T4]) following the rule described in [MST] after a modification, which is just such an alternation of the homology orientation which eliminates the sign $(-1)^{b(M,N)}$ that appears in the product formula of [MST]. In the other words, with such a homology orientation the product formula will look like A4, in §3 below.

    Note that such a homology orientation in the fiber sums is preserved by the natural diffeomorphisms $X\#_\Sigma Y\cong Y\#_\Sigma X$ and $(X\#_\Sigma Y)\#_\Sigma Z\cong X\#_\Sigma(Y\#_\Sigma Z)$. In the case of symplectic pairs (X,Σ) and (Y,Σ), the symplectic homology orientations in X and Y induce the symplectic homology orientation of $X\#_\Sigma Y$.

    If $\Sigma^2>0$, then we choose the homology orientation of $\hat X$ induced by that of X.

1.4. The properties of $S_{X,\Sigma}$.

1.4.1. Theorem. *The invariant $S_{X,\Sigma}\colon \mathrm{Spin}^{\mathbb C}(X,\Sigma)\to R_{X,\Sigma}$ is independent of the choice of W and has the following properties.*

(1) *Finiteness of the set $B_{X,\Sigma}=\{\mathfrak{r}\in \mathrm{Spin}^{\mathbb C}(X,\Sigma)\,|\,S_{X,\Sigma}(\mathfrak{r})\neq 0\}$.*
(2) *The blow-up relation $S_{\hat X,\Sigma}(\hat{\mathfrak{r}})=S_{X,\Sigma}(\mathfrak{r})$ (if $\Sigma^2>0$ in X).*
(3) *The conjugation symmetry $S_{X,\Sigma}=\pm S_{X,-\Sigma}\circ\mathrm{conj}$, where $-\Sigma$ is Σ with the opposite orientation, and $\mathrm{conj}\colon \mathrm{Spin}^{\mathbb C}(X,\Sigma)\to \mathrm{Spin}^{\mathbb C}(X,-\Sigma)$ the conjugation involution.*
(4) *Normalization: if X admits a relatively minimal Lefschetz pencil with a fiber Σ and is endowed with the canonical homology orientation of a symplectic manifold (for a symplectic structure compatible with the pencil), then there is only one basic relative $\mathrm{Spin}^{\mathbb C}$ structure, $\mathfrak{r}_{X,\Sigma}\in \mathrm{Spin}^{\mathbb C}(X,\Sigma)$ (the canonical structure of the pencil, see 2.6) and $S_{X,\Sigma}(\mathfrak{r}_{X,\Sigma})=1$.*
(5) *Splitting formula relating the absolute and the relative SW invariants:*

$$S_X(\mathfrak{s})=\sum_{r\in abs^{-1}_{X,\Sigma}(\mathfrak{s})} S_{X,\Sigma}(\mathfrak{r}),$$

for any $\mathfrak{s}\in \mathrm{Spin}^{\mathbb C}(X)$ such that $c_1(\mathfrak{s})[\Sigma]=\chi(\Sigma)+\Sigma^2$.
(6) *The product formula for a fiber sum $X=X_+\#_\Sigma X_-$ says:*

$$\underline{S}_{X,\Sigma}=(\underline{S}_{X_-,\Sigma})(\underline{S}_{X_+,\Sigma})$$

More explicitly, this means that for any $\mathfrak{r}\in \mathrm{Spin}^{\mathbb C}(X,\Sigma)$

$$S_{X,\Sigma}(\mathfrak{r})=\sum_{\mathfrak{r}_+\vee\mathfrak{r}_-=\mathfrak{r}} S_{X_+,\Sigma}(\mathfrak{r}_+)S_{X_-,\Sigma}(\mathfrak{r}_-)$$

where $\mathfrak{r}_\pm$ are varying in $\mathrm{Spin}^{\mathbb C}(X_\pm,\Sigma)$. Equivalently, one can write it as

$$S_{X,\Sigma}(\mathfrak{r}_+\vee\mathfrak{r}_-)=\sum_{k\in\mathbb{Z}} S_{X_+,\Sigma}(\mathfrak{r}_++k\sigma_+)S_{X_-,\Sigma}(\mathfrak{r}_--k\sigma_-)$$

where $\sigma_\pm \in H^2(X_\pm, \Sigma)$ is dual to the fundamental class of Σ shifted inside $X_\pm \smallsetminus \Sigma$.

(7) *Adjunction inequality for $\mathfrak{r} \in B_{X,\Sigma}$ and a membrane $F \subset X$, with the connected complement $\Sigma \smallsetminus \partial F$ (note that positivity of F^2 is not required)*

$$-\chi(F) \geqslant F^2 + |\mathfrak{r}[F]|.$$

The relations (4) and (5) of Theorem 1.4.1 imply together

Corollary 1.4.2. *For any $\mathfrak{s} \in \mathrm{Spin}^{\mathbb{C}}(X)$ such that $c_1(\mathfrak{s})[\Sigma] = \chi(\Sigma) + \Sigma^2$*

$$S_X(\mathfrak{s}) = \sum_{\mathfrak{r}_+ \#_\Sigma \mathfrak{r}_- = \mathfrak{s}} S_{X_+,\Sigma}(\mathfrak{r}_+) S_{X_-,\Sigma}(\mathfrak{r}_-)$$

Remarks.

(1) The sign "$\pm$" in Theorem 1.4.1(3) is $(-1)^{\frac{1}{4}(\sigma(X)+\chi(X))}$, like for the absolute invariant S_X.

(2) The product of $\underline{S}_{X_\pm,\Sigma}$ in Theorem 1.4.1(6), is induced by the natural affine map

$$\mathrm{Spin}^{\mathbb{C}}(X_+, \Sigma) \times \mathrm{Spin}^{\mathbb{C}}(X_-, \Sigma) \cong \mathrm{Spin}^{\mathbb{C}}(X, M) \to \mathrm{Spin}^{\mathbb{C}}(X, \Sigma)$$

associated with the corresponding cohomology homomorphisms.

(3) If $b_2^+(X) = 1$, then the splitting formula 1.4.1(5) should be applied to $SW_X(\mathfrak{s})$ defined by fixing the $[\Sigma]$-positive orientation of the line $H_2^+(X)$, and $OS_X(\mathfrak{s})$ is defined by fixing the line spanned by $[\Sigma]$ (because these are the choices involved into the corresponding product formulae).

(4) Theorem 1.4.1 does not mention some more straightforward properties, which do not involve Σ , for example, the adjunction inequality for a closed surface F, the blowup formula at a point $x \notin \Sigma$, and the product formula with respect to an additional surface Σ' in $X_\pm$ disjoint from Σ. All these properties are formulated exactly like in the case of the absolute invariants S_X and proved by giving an obvious reference to the case of absolute invariants of the corresponding fiber sums.

(5) The invariant $S_{X,\Sigma}$ can be defined similarly for a multi-component surface Σ, as one can take fiber sums with auxiliary Lefschetz fibrations along all the components of Σ. The properties of such invariants are analogous to those formulated in Theorem 1.4.1, and the proofs just repeat the arguments in §4.

(6) The case of genus $g = 1$ is a bit special, mainly because the product formula in this case looks different. Nevertheless, the same definition for $S_{X,\Sigma}$ can be given for $g = 1$, and all the properties except the splitting formula (5) in Theorem 1.4.1 still hold. In the case of $S_X = SW_X$, this follows from the results of Taubes [T5], except for the property (7) (not discussed in [T5]), which is proved by the same arguments as in the case $g > 1$.

1.5. Application: the genus estimate for membranes. By definition, a membrane on a surface Σ in X is a compact surface $F \subset X$ with the boundary $\partial F = F \cap \Sigma$, at no point of which F is tangent to Σ. The self-intersection index F^2 is defined with respect to the normal framing along ∂F which is tangent to Σ. The number $\mathfrak{r}[F]$ (evaluation of $\mathfrak{r} \in \mathrm{Spin}^{\mathbb{C}}(X, \Sigma)$ on F) is defined in 2.6. Throughout the paper we suppose that membranes are connected and oriented, although the adjunction inequality holds as well for disconnected membranes, which follows from additivity of $\chi(F)$, F^2 and $\mathfrak{r}[F]$.

The adjunction inequality for membranes implies for instance the minimal genus property for symplectic and Lagrangian membranes in symplectic manifolds, namely

1.5.1. Corollary. *Assume that X is a symplectic 4-manifold, $\Sigma \subset X$ is an essential surface with $\Sigma^2 \geqslant 0$, $g(\Sigma) > 1$, and $F \subset X$ is a membrane on Σ with the connected complement $\Sigma \smallsetminus L$ of the boundary $L = \partial F$. Assume furthermore that either Σ is symplectic and F is Lagrangian, or vice versa, Σ is Lagrangian and F is symplectic. Then for any membrane, $F' \subset X$, which has the same boundary $L = \partial F'$, defines the same normal framing for Σ along each component of L, and realizes the same class $[F', \partial F'] = [F, \partial F] \in H_2(X, \Sigma)$ we have $g(F') \geqslant g(F)$.*

Proof. The adjunction inequality 1.4.1(7) becomes an equality for such X, Σ, F, and for the canonical $\mathrm{Spin}^{\mathbb{C}}$ structure $\mathfrak{r} \in \mathrm{Spin}^{\mathbb{C}}(X, \Sigma)$ (symplectic or Lagrangian, depending on the case considered). The assumptions formulated for F' imply that $(F')^2 = F^2$ and $\mathfrak{r}[F'] = \mathfrak{r}[F]$, so 1.4.1(7) yields the required estimate for $g(F')$. $\square$

Another example of application of 1.4.1(7) is orthogonality of the relative basic classes, $\mathfrak{r} \in B_{X,\Sigma}$, to the (-1)-disc membranes.

1.5.2. Corollary. *If $D \subset X$ is a (-1)-disc membrane on a surface $\Sigma \subset X$ as above, then $\mathfrak{r}[D] = 0$ for all $\mathfrak{r} \in B_{X,\Sigma}$.* $\square$

1.6. On the calculation of the invariants $S_{X,\Sigma}$. The first observation concerns vanishing of the relative invariants $S_{X,\Sigma}$ if Σ is not a minimal genus surface in its homology class, because non-vanishing would contradict to the adjunction inequality applied in $X \#_\Sigma W$ to the surface Σ' homologous to Σ but of a smaller genus. This argument does not work if Σ' cannot be made disjoint from Σ in X (although the author does not know such examples, in which it really cannot).

Another example of calculation of $S_{X,\Sigma}$ is contained in Theorem 1.4.1(4). It is an interesting question if the normalization property 1.4.1(4) holds as well in a more general setting, namely for symplectic relatively minimal pairs (X, Σ), such that $[\Sigma] \in H_2(X)$ is a primitive class.

One more important example of calculation of $SW_{X,\Sigma}$ can be extracted from the work [FS2], where these invariants appeared to distinguish the embeddings of surfaces obtained by the rim-surgery from Σ. In fact, the results of [FS2] mean that the invariant $SW_{X,\Sigma}$ is multiplied by the Alexander polynomial after performing a rim knot surgery. More precisely, assume that $\ell \subset \Sigma$ is a simple closed curve, $K \subset S^3$ is a knot and $\Sigma_{K,\ell} \subset X$ is a surface obtained from $\Sigma \subset X$ by rim surgery along ℓ using K as a pattern.

1.6.1. Theorem. $\underline{SW}_{X,\Sigma_{K,\ell}} = \Delta_K(\delta([\ell]^*))\underline{SW}_{X,\Sigma}$. $\square$

Here $[\ell]^* \in H^1(\Sigma)$ is dual to $[\ell] \in H_1(\Sigma)$, δ is the boundary map $H^1(\Sigma) \to H^2(X, \Sigma)$, Δ_K is the Alexander polynomial in the symmetrized form, and $\Delta_K(\delta([\ell]^*))$ is considered as an element of the group ring $\mathbb{Z}[H^2(X, \Sigma)] \subset R_{X,\Sigma}[H^2(X, \Sigma)]$.

1.7. The structure of the paper. In §2 we give a brief summary on the absolute $\mathrm{Spin}^{\mathbb{C}}$ structures and develop some calculus of the relative structures that is used in §4.

In §3, we recall some fundamental properties of SW and OS invariants that are required to construct their relative versions. Although mostly well-known, these properties appear in literature in various settings, not always in the form convenient for us, so we give some comments and references. In the core section, §4, we prove Theorem 1.4.1.

In §5, we discuss some generalizations of the invariant $S_{X,\Sigma}$. Its version, $S_{X,\Sigma,K}$, discussed in 5.1, depends on a subgroup $K \subset H^1(\Sigma)$, which suggests an analogy with the invariant in [CW]. A generalization of $S_{X,\Sigma}$ in 5.2, which we presented for simplicity only

in the case of OS invariants, is a relative version of the invariant $F^{\mathrm{mix}}_{X,\mathfrak{s}}$ from [OS3] (here instead of relativization with respect to a surface $\Sigma \subset X$ we consider relativization with respect to a boundary component of X).

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§2. Absolute and relative $\mathrm{Spin}^{\mathbb{C}}$ structures

2.1. Absolute $\mathrm{Spin}^{\mathbb{C}}$ structures. A $\mathrm{Spin}^{\mathbb{C}}$ structure in a principle SO_n bundle, $P \to X$, is an isomorphism class of $\mathrm{Spin}^{\mathbb{C}}_n$-extensions, $S \to P$, of P. The set of $\mathrm{Spin}^{\mathbb{C}}$ structures, $\mathrm{Spin}^{\mathbb{C}}(P)$, has a natural action $\mathrm{Spin}^{\mathbb{C}}(P) \times H^2(X) \to \mathrm{Spin}^{\mathbb{C}}(P)$, $(\mathfrak{s}, h) \mapsto \mathfrak{s} + h$, which makes it an affine space over $H^2(X)$. The projection $\mathrm{Spin}^{\mathbb{C}}_n \to \mathrm{SO}_n \times U_1 \to U_1$ associates to S its determinant U_1-bundle, $\det S$, with the Chern class $c_1(S) = c_1(\det S)$, so that $c_1(\mathfrak{s} + h) = c_1(\mathfrak{s}) + 2h$.

We simplify the notation writing just $\mathrm{Spin}^{\mathbb{C}}(X)$ instead of $\mathrm{Spin}^{\mathbb{C}}(P)$, if a principal bundle $P \to X$ is associated with an obvious vector bundle $E \to X$, for example, with the tangent bundle of a manifold X (the choice of the euclidian structure in E is not essential).

2.2. The conjugation involution. The conjugate, $\overline{S} \to X$, to a principal $\mathrm{Spin}^{\mathbb{C}}_n$ bundle, $S \to X$, set-theoretically coincides with the latter, but has the conjugate action of $\mathrm{Spin}^{\mathbb{C}}_n$ (induced by the conjugation automorphism in $\mathrm{Spin}^{\mathbb{C}}_n$, which covers the direct product automorphism of $\mathrm{SO}_n \times U_1$, identical on SO_n and non-identical in U_1). The conjugation defines an involution, $\mathrm{conj}_P \colon \mathrm{Spin}^{\mathbb{C}}(P) \to \mathrm{Spin}^{\mathbb{C}}(P)$, $\mathfrak{s} \mapsto \bar{\mathfrak{s}}$, such that $c_1(\bar{\mathfrak{s}}) = -c_1(\mathfrak{s})$ and $\overline{\mathfrak{s} + h} = \bar{\mathfrak{s}} - h$ for any $\mathfrak{s} \in \mathrm{Spin}^{\mathbb{C}}(P)$ and $h \in H^2(X)$.

2.3. Homology interpretation of $\mathrm{Spin}^{\mathbb{C}}$ structures. It is convenient to identify the set $\mathrm{Spin}^{\mathbb{C}}(P)$ with the coset of the image of $H^2(X)$ under the monomorphism $\pi_P^* \colon H^2(X) \to H^2(P)$. This makes transparent the nature of the affine structure in $\mathrm{Spin}^{\mathbb{C}}(P)$. Namely, a $\mathrm{Spin}^{\mathbb{C}}$ extension $F \colon S \to P$ can be viewed as a principal U_1-bundle over P since $\ker(\mathrm{Spin}^{\mathbb{C}}_n \to \mathrm{SO}_n) \cong U_1$, and the Chern class $c_1(F) \in H^2(P)$ defines the correspondence between $\mathrm{Spin}^{\mathbb{C}}$ structures and those cohomology classes which have a non-trivial restriction $H^2(P) \to H^2(\mathrm{SO}_n) \cong \mathbb{Z}/2$, $n \geqslant 3$, to a fiber of P.

Given $\mathfrak{s} \in \mathrm{Spin}^{\mathbb{C}}(P) \subset H^2(P)$, one can observe that $\pi_P^*(c_1(\mathfrak{s})) = 2\mathfrak{s}$ and that $\bar{\mathfrak{s}} = -\mathfrak{s}$.

2.4. The canonical $\mathrm{Spin}^{\mathbb{C}}$ structure of a Lefschetz fibration. An almost complex structure in a SO_{2n} bundle defines the $\mathrm{Spin}^{\mathbb{C}}$-extension associated to the natural homomorphism $U_n \to \mathrm{Spin}^{\mathbb{C}}_n$. In particular, a symplectic manifold carries a canonical $\mathrm{Spin}^{\mathbb{C}}$ structure represented by *the symplectic* $\mathrm{Spin}^{\mathbb{C}}$ *extension*, $S \to X$. It is well-known that the total space X of a Lefschetz fibration $p \colon X \to S^2$, carries a compatible symplectic structure (except the case of null-homologous fibers of genus 1, in which X is still almost complex) which gives the associated canonical $\mathrm{Spin}^{\mathbb{C}}$ structure.

In fact, to define a $\mathrm{Spin}^{\mathbb{C}}$ structure in a vector bundle $E \to X$ it is sufficient to have an almost complex structure over its 3-skeleton, $\mathrm{Ske}_3 X$, only (more precisely, $\mathrm{Spin}^{\mathbb{C}}$ structures can be viewed as equivalence classes of those almost complex structures over

$\mathrm{Ske}_2\, X$ that can be extended to $\mathrm{Ske}_3\, X$). This gives an alternative way to introduce the canonical $\mathrm{Spin}^{\mathbb{C}}$ structure in the Lefschetz fibration, using $U_1 \times U_1$ reduction of the tangent bundle τ_X in the complement of the critical point set of p determined by the "vertical" and the complementary "horizontal" $\mathrm{SO}_2 = U_1$ subbundles.

2.5. Relative $\mathrm{Spin}^{\mathbb{C}}$-structures. In the definition of a relative structure, in addition to the $\mathrm{Spin}^{\mathbb{C}}_n$-bundle $S \to X$ considered in 2.1, we fix an isomorphism between the restriction $S|_A$ and a certain reference principal $\mathrm{Spin}^{\mathbb{C}}$ bundle, $S_A \to A$. Such a reference bundle appears naturally for example if X is a 4-manifold and A is a surface $\Sigma \subset X$, or a tubular neighborhood N of Σ, or the boundary ∂N, since in these cases $\tau_X|_A$ admits a natural $U_1 \times U_1$ reduction and thus, the associated $\mathrm{Spin}^{\mathbb{C}}_4$-extension.

More formally speaking, let (X, A) be a CW-pair, $\pi_P \colon P \to X$ a principal SO_n bundle, $\pi_{P|A} \colon P_A \to A$ the restriction of π_P over A, and $F_A \colon S_A \to P_A$ a $\mathrm{Spin}^{\mathbb{C}}$ extension of P_A. *A relative* $\mathrm{Spin}^{\mathbb{C}}$ *extension of* P with respect to S_A is a $\mathrm{Spin}^{\mathbb{C}}$-extension, $S \to X$, $F \colon S \to P$, together with an isomorphism $R \colon S|_A \to S_A$, such that $F_A \circ R$ is the restriction of F to A. An isomorphism between relative $\mathrm{Spin}^{\mathbb{C}}$-extensions $F^{(i)} \colon S^{(i)} \to P$, $R^{(i)} \colon S^{(i)}|_A \to S_A$, $i = 1, 2$, is defined as an isomorphism $S^{(1)} \to S^{(2)}$ of $\mathrm{Spin}^{\mathbb{C}}$ bundles whose restriction over A commutes with $R^{(1)}$ and $R^{(2)}$. An isomorphism class of relative $\mathrm{Spin}^{\mathbb{C}}$-extensions is called *a relative* $\mathrm{Spin}^{\mathbb{C}}$ *structure*, and the set of such structures is denoted by $\mathrm{Spin}^{\mathbb{C}}(P, S_A)$, or simply by $\mathrm{Spin}^{\mathbb{C}}(X, A)$ if P and S_A are evident.

It is straightforward to check that $\mathrm{Spin}^{\mathbb{C}}(P, S_A)$ is an affine space over $H^2(X, A)$ and the natural forgetful map *abs*: $\mathrm{Spin}^{\mathbb{C}}(P, S_A) \to \mathrm{Spin}^{\mathbb{C}}(P)$ is affine with respect to the cohomology forgetful homomorphism $H^2(X, A) \to H^2(X)$.

The conjugation involution defined like in the absolute case interchanges $\mathrm{Spin}^{\mathbb{C}}(P, S_A)$ with $\mathrm{Spin}^{\mathbb{C}}(P, \overline{S}_A)$. It is anti-affine, that is $\overline{\mathfrak{r} + h} = \overline{\mathfrak{r}} - h$ for $\mathfrak{r} \in \mathrm{Spin}^{\mathbb{C}}(P, S_A)$, $h \in H^2(X, A)$.

2.6. Relative $\mathrm{Spin}^{\mathbb{C}}$ structures with respect to surfaces, $\Sigma \subset X$, and their evaluation on membranes. Let $S_\Sigma \to \Sigma$ denote the canonical $\mathrm{Spin}^{\mathbb{C}}$ extension defined by the $U_1 \times U_1$ reduction due to the splitting of the tangent bundle $\tau_X|_\Sigma$ along Σ into a sum $\tau_\Sigma \oplus \nu_\Sigma$ of the tangent and the normal bundles to Σ. Note that the inversion of the orientation of Σ results in the conjugation of the associated canonical $\mathrm{Spin}^{\mathbb{C}}_4$ bundle, $S_{-\Sigma} = \overline{S}_\Sigma$. In particular, the conjugation involution in this case is $\mathrm{Spin}^{\mathbb{C}}(X, \Sigma) \to \mathrm{Spin}^{\mathbb{C}}(X, -\Sigma)$.

Assume that $\mathfrak{r} \in \mathrm{Spin}^{\mathbb{C}}(X, \Sigma)$. Note that any membrane $(F, \partial F) \subset (X, \Sigma)$ defines trivializations of the both τ_Σ and ν_Σ along ∂F, and thus provides a trivialization of the determinant bundle $\det S_\Sigma \cong \tau_\Sigma \otimes \nu_\Sigma$. The obstruction class in $H^2(F, \partial F)$ for extension of this trivialization to the whole F, as it is evaluated on the fundamental class, $[F, \partial F]$, gives an integer denoted by $\mathfrak{r}[F]$. It is easy to observe that $\overline{\mathfrak{r}}[F] = \mathfrak{r}[-F] = -(\mathfrak{r}[F])$.

2.7. The canonical relative $\mathrm{Spin}^{\mathbb{C}}$ structures in the case of symplectic or Lagrangian surface, $\Sigma \subset X$. Assume now that $\Sigma \subset X$ is a symplectic surface with respect to some symplectic structure ω in X that is $\omega|_\Sigma > 0$. Then we can define the canonical *symplectic relative* $\mathrm{Spin}^{\mathbb{C}}$ *structure*, $\mathfrak{r}_{X,\Sigma} \in \mathrm{Spin}^{\mathbb{C}}(X, \Sigma)$ whose image $abs(\mathfrak{r}_{X,\Sigma}) \in \mathrm{Spin}^{\mathbb{C}}(X)$ is the absolute symplectic canonical $\mathrm{Spin}^{\mathbb{C}}$ structure introduced in section 2.4. Namely, the structure $\mathfrak{r}_{X,\Sigma}$ is represented by a $\mathrm{Spin}^{\mathbb{C}}$ extension $S \to P$ of the principal SO_4 bundle $P \to X$, which arises from an almost complex structure determined in τ_X as we fix a Riemannian metric in X compatible with ω. If we choose such a metric making

the surface Σ pseudo-holomorphic (which is always possible), then the restriction $S|_\Sigma$ is naturally identified with the canonical $\mathrm{Spin}^{\mathbb{C}}$-bundle $S_\Sigma \to \Sigma$.

Consider now the case of a Lagrangian surface Σ in a symplectic manifold X, in which we can similarly define the canonical *Lagrangian relative* $\mathrm{Spin}^{\mathbb{C}}$ *structure*, $\mathfrak{r}_{X,\Sigma} \in \mathrm{Spin}^{\mathbb{C}}(X,\Sigma)$. One way to do it is to make a Lagrangian surface symplectic by a perturbation. There is also an alternative description of this structure (which concerns also the case of a null-homologous torus in which such a perturbation is impossible, and makes evident that the choice of a perturbation is not essential), in which we use the canonical isomorphism between the two different almost complex structure in $\tau_X|_\Sigma$: the one induced from X that is coming from the isomorphism $\tau_X|_\Sigma \cong \tau_\Sigma \otimes \mathbb{C}$, and the one arising from $U_1 \times U_1$ reduction due to the splitting $\tau_X|_\Sigma \cong \tau_\Sigma \oplus \nu_\Sigma$. It is just a special case of the canonical isomorphism $\xi \otimes \mathbb{C} \cong \xi \oplus \bar{\xi}$ for a complex bundle ξ. The induced isomorphism of the associated $\mathrm{Spin}^{\mathbb{C}}$ bundles covers an automorphism of $\tau_X|_\Sigma$ that can be canonically connected to the identity by an isotopy. This gives an isomorphism between these $\mathrm{Spin}^{\mathbb{C}}$ extensions, which defines the Lagrangian relative $\mathrm{Spin}^{\mathbb{C}}$ structure.

Remark. Note that the canonical $\mathrm{Spin}^{\mathbb{C}}$ structure $\mathfrak{r}_{X,\Sigma} \in \mathrm{Spin}^{\mathbb{C}}(X,\Sigma)$ of a symplectic pair (X,Σ) is invariant under the monodromy induced by any symplectic isotopy of Σ in X, whereas any other structure, $\mathfrak{r} = \mathfrak{r}_{X,\Sigma} + h \in \mathrm{Spin}^{\mathbb{C}}(X,\Sigma)$, $h \in H^2(X,\Sigma)$, is sent by the monodromy to $\mathfrak{r}_{X,\Sigma} + f_*(h)$, where f_* is the cohomology monodromy.

The same concerns Lagrangian surfaces and Lagrangian isotopy.

2.8. Lefschetz fibrations and their conjugates. A special case of our interest is Σ being a fiber of a Lefschetz fibration $p\colon X \to S^2$, (or more generally, a fiber in a Lefschetz pencil). Such a fiber $\Sigma \subset X$ is symplectic with respect to the symplectic form ω supported by the Lefschetz fibration (or pencil), so there is a canonical structure $\mathfrak{r}_{X,\Sigma} \in \mathrm{Spin}^{\mathbb{C}}(X,\Sigma)$ from section 2.7.

The conjugate Lefschetz fibration $\overline{p}\colon \overline{X} \to \overline{S^2}$ is by definition, set-theoretically the same as p, however, with the opposite orientation chosen in the base-space $\overline{S^2} = -S^2$ and in the fibers, $\overline{\Sigma} = -\Sigma$ (so that $\overline{X}$ itself has the same orientation as X). It is not difficult to observe that $\overline{\mathfrak{r}_{X,\Sigma}} = \mathfrak{r}_{\overline{X},\overline{\Sigma}} \in \mathrm{Spin}^{\mathbb{C}}(X,-\Sigma)$.

2.9. The excision and the homotopy invariance theorems for $\mathrm{Spin}^{\mathbb{C}}$ structures. The propositions stated below mimic the standard results for the cohomology and follow automatically from the latters, since an affine map associated with an isomorphism must be an affine isomorphism.

2.9.1. Proposition (excision). *Assume that a CW complex Z is decomposed into a union of subcomplexes, $Z = X \cup Y$, $A = X \cap Y$. Consider a principal SO_n bundle $P_Z \to Z$ and let P_X, P_Y, P_A denote its restrictions over X, Y, and A respectively. Fix a $\mathrm{Spin}^{\mathbb{C}}$ extension, $F_Y\colon S_Y \to P_Y$ and let $F_A\colon S_A \to P_A$ denote its restriction over A. Then the restriction map*

$$\mathrm{Spin}^{\mathbb{C}}(P_Z, S_Y) \to \mathrm{Spin}^{\mathbb{C}}(P_X, S_A)$$

is an isomorphism of affine spaces agreeing with the isomorphism $H^2(Z,Y) \cong H^2(X,A)$. $\square$

2.9.2. Proposition (homotopy invariance). *Assume that Σ is a deformation retract of $N \subset X$. Let $F_N\colon S_N \to P_N$ be a $\mathrm{Spin}^{\mathbb{C}}$ extension and $S_\Sigma = S_N|_\Sigma$. Then the restriction map $\mathrm{Spin}^{\mathbb{C}}(P_X, S_N) \to \mathrm{Spin}^{\mathbb{C}}(P_X, S_\Sigma)$ is an isomorphism of the affine spaces agreeing with the isomorphism $H^2(X,N) \cong H^2(X,\Sigma)$.* $\square$

2.9.3. Corollary. *Let $\Sigma \subset X$ be a surface in a 4-manifold, $N \subset X$ its compact tubular neighborhood, $M = \partial N$, and $X^\circ = X \smallsetminus \mathrm{Int}(N)$. Then we have canonical affine isomorphisms* $\mathrm{Spin}^{\mathbb{C}}(X, \Sigma) \cong \mathrm{Spin}^{\mathbb{C}}(X, N) \cong \mathrm{Spin}^{\mathbb{C}}(X^\circ, M)$. $\square$

2.10. Connected sums and blowing up of $\mathrm{Spin}^{\mathbb{C}}$ **structures.** Definitions of the connected sum and the blowup operations are obvious and well-known for the absolute $\mathrm{Spin}^{\mathbb{C}}$ structures. They can be easily extended to relative $\mathrm{Spin}^{\mathbb{C}}$ structures, as well. For given n-manifolds $X_\pm$, with codimension 2 submanifolds $\Sigma_\pm$, and $\mathfrak{r}_\pm \in \mathrm{Spin}^{\mathbb{C}}(X_\pm, \Sigma_\pm)$, we obtain the connected sum $\mathfrak{r}_+ \# \mathfrak{r}_- \in \mathrm{Spin}^{\mathbb{C}}(X_+ \# X_-, \Sigma_+ \# \Sigma_-)$, where $\Sigma_+ \# \Sigma_- \subset X_+ \# X_-$ is the internal connected sum of $(X_\pm, \Sigma_\pm)$.

Let $\Sigma \subset X$ be a surface in a four-manifold, and $\hat\Sigma \subset \hat X$ its proper image after blowing up X at a point of Σ, that is $\Sigma \# \mathbb{C}\mathrm{P}^1 \subset X \# (-\mathbb{C}\mathrm{P}^2)$. For $\mathfrak{r} \in \mathrm{Spin}^{\mathbb{C}}(X, \Sigma)$, we define $\hat{\mathfrak{r}} \in \mathrm{Spin}^{\mathbb{C}}(\hat X, \hat \Sigma)$ as $\hat{\mathfrak{r}} = \mathfrak{r} \# \mathfrak{r}_{-1}$, where $\mathfrak{r}_{-1} \in \mathrm{Spin}^{\mathbb{C}}(-\mathbb{C}\mathrm{P}^2, \mathbb{C}\mathrm{P}^1)$ is the unique structure such that $c_1(abs(\mathfrak{r}_{-1})) = -1$.

2.11. Fiber sums of relative $\mathrm{Spin}^{\mathbb{C}}$ **structures.** Let Σ be a closed oriented surface of genus $g > 1$. We say that X is *a Σ-marked 4-manifold*, if there is a fixed smooth embedding $f \colon \Sigma \to X$ endowed with a normal framing of $f(\Sigma)$ (in particular, $\Sigma^2 = 0$). To simplify the notation, we will be writing $\mathrm{Spin}^{\mathbb{C}}(X, \Sigma)$ rather then $\mathrm{Spin}^{\mathbb{C}}(X, f(\Sigma))$.

Given Σ-marked 4-manifolds $X_\pm$, consider their fiber sum $X = X_+ \#_\Sigma X_- = X_+^\circ \cup_f X_-^\circ$, where $X_\pm^\circ = X_\pm \smallsetminus \mathrm{Int}(N_\pm)$ (the complements of the tubular neighborhoods of Σ), and the gluing diffeomorphism $f \colon \partial N_+ \to \partial N_-$ is naturally determined by the trivialization of $N_\pm \to \Sigma$ respecting the framings, so that ∂N_+ and $-\partial N_-$ are identified with $M = \Sigma \times S^1$. Note that X has an induced structure of Σ-marked 4-manifold, since $\Sigma_t = \Sigma \times t \subset \Sigma \times S^1$ has a natural normal framing.

Operations $\mathfrak{r}_+ \#_\Sigma \mathfrak{r}_- \in \mathrm{Spin}^{\mathbb{C}}(X)$ and $\mathfrak{r}_+ \vee \mathfrak{r}_- \in \mathrm{Spin}^{\mathbb{C}}(X, \Sigma)$ for $\mathfrak{r}_\pm \in \mathrm{Spin}^{\mathbb{C}}(X_\pm, \Sigma)$ are the compositions of the isomorphism

$$\mathrm{Spin}^{\mathbb{C}}(X_+, \Sigma) \times \mathrm{Spin}^{\mathbb{C}}(X_-, \Sigma) \cong \mathrm{Spin}^{\mathbb{C}}(X_+^\circ, M) \times \mathrm{Spin}^{\mathbb{C}}(X_-^\circ, M) \cong \mathrm{Spin}^{\mathbb{C}}(X, M)$$

with the forgetful maps $\mathrm{Spin}^{\mathbb{C}}(X, M) \to \mathrm{Spin}^{\mathbb{C}}(X)$ and $\mathrm{Spin}^{\mathbb{C}}(X, M) \to \mathrm{Spin}^{\mathbb{C}}(X, \Sigma)$.

Given $\mathfrak{s}_\pm \in \mathrm{Spin}^{\mathbb{C}}(X_\pm, \mathfrak{s}_\Sigma)$ we denote by $\mathfrak{s}_- \#_\Sigma \mathfrak{s}_+$ a subset of $\mathrm{Spin}^{\mathbb{C}}(X)$ consisting of the structures $\mathfrak{r}_- \#_\Sigma \mathfrak{r}_+$ for all $\mathfrak{r}_\pm \in abs^{-1}_{X_\pm, \Sigma}(\mathfrak{s}_\pm)$. It is not difficult to check that the set $\mathfrak{s}_- \#_\Sigma \mathfrak{s}_+$ is affine with respect to the subgroup $\Delta_M \subset H^2(X)$, which is the image of $H^1(\Sigma)$ under the product of the homomorphism $q^* \colon H^1(\Sigma) \to H^1(M)$ induced by the projection $q \colon M \cong \Sigma \times S^1 \to \Sigma$ and the boundary map $\delta_M \colon H^1(M) \to H^2(X)$.

One can also interpret $\mathfrak{s}_- \#_\Sigma \mathfrak{s}_+$ as a set consisting of those $\mathfrak{s} \in \mathrm{Spin}^{\mathbb{C}}(X)$ which have $d(\mathfrak{s}) = d(\mathfrak{s}_+) + d(\mathfrak{s}_-)$ and whose restriction to $X_\pm^\circ$ coincides with that of $\mathfrak{s}_\pm$.

2.12. The natural properties of the operations with the relative $\mathrm{Spin}^{\mathbb{C}}$ **structures.** It is not difficult to check that the operations introduced in 2.11 satisfy the following natural properties

$$\mathfrak{r}_1 \vee \mathfrak{r}_2 = \mathfrak{r}_2 \vee \mathfrak{r}_1, \text{ and thus } \mathfrak{r}_1 \#_\Sigma \mathfrak{r}_2 = \mathfrak{r}_2 \#_\Sigma \mathfrak{r}_1$$
$$(\mathfrak{r}_1 \vee \mathfrak{r}_2) \vee \mathfrak{r}_3 = \mathfrak{r}_1 \vee (\mathfrak{r}_2 \vee \mathfrak{r}_3), \text{ and thus } (\mathfrak{r}_1 \vee \mathfrak{r}_2) \#_\Sigma \mathfrak{r}_3 = \mathfrak{r}_1 \#_\Sigma (\mathfrak{r}_2 \vee \mathfrak{r}_3)$$
$$\overline{\mathfrak{r}_1 \vee \mathfrak{r}_2} = \bar{\mathfrak{r}}_1 \vee \bar{\mathfrak{r}}_2, \text{ and thus } \overline{\mathfrak{r}_1 \#_\Sigma \mathfrak{r}_2} = \bar{\mathfrak{r}}_1 \#_{-\Sigma} \bar{\mathfrak{r}}_2$$

where the equalities mean that the obvious diffeomorphisms

$$(X_1\#_\Sigma X_2,\Sigma)\cong(X_2\#_\Sigma X_1,\Sigma)$$
$$((X_1\#_\Sigma X_2)\#_\Sigma X_3,\Sigma)\cong(X_1\#_\Sigma(X_2\#_\Sigma X_3),\Sigma)$$
$$X_1\#_\Sigma X_2\cong X_1\#_{-\Sigma}X_2$$

send one of the corresponding $\mathrm{Spin}^{\mathbb{C}}$ structures to the other.

Some ambiguity in the notion of "the obvious diffeomorphism", related in particular to the ambiguity in Σ-marking of the fiber sums, turns out to be not essential. It is also straightforward to check the following

Proposition 2.12.1. *Assume that (X_i,Σ) are symplectic pairs, $i=1,2$, $X=X_1\#_\Sigma X_2$, and $\mathfrak{r}_i\in\mathrm{Spin}^{\mathbb{C}}(X_i,\Sigma)$ are the canonical relative $\mathrm{Spin}^{\mathbb{C}}$ structures. Then $\mathfrak{r}_1\vee\mathfrak{r}_2\in\mathrm{Spin}^{\mathbb{C}}(X,\Sigma)$ is also the canonical $\mathrm{Spin}^{\mathbb{C}}$ structure of the symplectic pair (X,Σ). In particular, $\mathfrak{r}_1\#_\Sigma\mathfrak{r}_2\in\mathrm{Spin}^{\mathbb{C}}(X)$ is the canonical symplectic $\mathrm{Spin}^{\mathbb{C}}$ structure of X.* □

§3. The basic properties of the absolute SW and OS invariants

3.1. The axioms. Axioms A1, A3 and A4 below are essential for the definition of $S_{X,\Sigma}$, for showing its independence from W, whereas axioms A2 and A5 are required only for proving the corresponding properties of $S_{X,\Sigma}$, namely, (3) and (7) in Theorem 1.4.1. Unless it is stated otherwise, we suppose in this section that all the closed 4-manifolds below have $b_2^+>1$ (in the case $b_2^+=1$ the formulations are similar, but require a bit more care).

A1. *Finiteness.* The set of the basic $\mathrm{Spin}^{\mathbb{C}}$ structures $B_X=\{\mathfrak{s}\in\mathrm{Spin}^{\mathbb{C}}(X)\,|\,S_X(\mathfrak{s})\neq 0\}$ is finite for any X.

A2. *Conjugation symmetry.* $S_X\circ\mathrm{conj}_X=\pm S_X$, where conj_X is the conjugation involution in $\mathrm{Spin}^{\mathbb{C}}(X)$.

A3. *Lefschetz normalization.* Assume that $X\to S^2$ is a relatively minimal Lefschetz fibration whose fiber $\Sigma\subset X$ has genus $g>1$. Let $\mathfrak{s}_X\in\mathrm{Spin}^{\mathbb{C}}(X)$ denote the canonical $\mathrm{Spin}^{\mathbb{C}}$ structure. Then

(1) $S_X(\mathfrak{s}_X)=1$, if X is endowed with the canonical homology orientation (with respect to a symplectic structure supporting the Lefschetz fibration).
(2) $\mathfrak{s}_X$ is the only basic structure $\mathfrak{s}\in\mathrm{Spin}^{\mathbb{C}}(X)$ satisfying the condition $c_1(\mathfrak{s})[\Sigma]=\chi(\Sigma)+\Sigma^2$;
(3) for any fiber sum $X\#_\Sigma Y=X^\circ\cup Y^\circ$ with a Σ-marked 4-manifold, the restriction $\mathfrak{s}|_{X^\circ}\in\mathrm{Spin}^{\mathbb{C}}(X^\circ)$ of any basic $\mathrm{Spin}^{\mathbb{C}}$ structure $\mathfrak{s}\in\mathrm{Spin}^{\mathbb{C}}(X\#_\Sigma Y,\mathfrak{s}_\Sigma)$, coincides with the restriction $\mathfrak{s}_X|_{X^\circ}\in\mathrm{Spin}^{\mathbb{C}}(X^\circ)$.

A4. *Product formula.* Let $X=X_-\#_\Sigma X_+=X_-^\circ\cup X_+^\circ$ be a fiber sum like in 2.11, with a fiber Σ of genus $g>1$. Choose $\mathfrak{s}_\pm\in\mathrm{Spin}^{\mathbb{C}}(X_\pm,\mathfrak{s}_\Sigma)$ and let $\sigma_\pm\in H^2(X_\pm)$ denote the Poincare dual class to $\Sigma\subset X_\pm$. Then

$$\sum_{k\in\mathbb{Z}}S_{X_-}(\mathfrak{s}_--k\sigma_-)S_{X_+}(\mathfrak{s}_++k\sigma_+)=\sum_{\mathfrak{s}\in\mathfrak{s}_-\#_\Sigma\mathfrak{s}_+}S_X(\mathfrak{s}).$$

A5. *Adjunction inequality:* $-\chi(\Sigma)\geqslant\Sigma^2+|c_1(\mathfrak{s})[\Sigma]|$, for any $\mathfrak{s}\in B_X$ and an essential surface $\Sigma\subset X$ of genus $g>0$, with $\Sigma^2\geqslant 0$.

Combining properties A3(1) and A4 and taking into account the remark about $\mathfrak{s}_-\#_\Sigma\mathfrak{s}_+$ and Δ_M in the end of section 2.11, we obtain the following

3.1.1. Corollary. *If the summands, X_-, involved into a fiber sum in A4 is a Lefschetz fibration with a fiber Σ, then for any $\mathfrak{s}_+ \in \mathrm{Spin}^{\mathbb{C}}(X_+, \mathfrak{s}_\Sigma)$*

$$S_{X_+}(\mathfrak{s}_+) = \sum_{\mathfrak{s} \in \mathfrak{s}_- \#_\Sigma \mathfrak{s}_+} S_X(\mathfrak{s}) = \sum_{\mathfrak{r}_+ \in abs^{-1}(\mathfrak{s}_+)} S_X(\mathfrak{r}_+ \#_\Sigma \mathfrak{r}_-) = \sum_{h \in \Delta_M} S_X(\mathfrak{s} + h)$$

where $\mathfrak{s}_- \in \mathrm{Spin}^{\mathbb{C}}(X_-)$, $\mathfrak{r}_- \in \mathrm{Spin}^{\mathbb{C}}(X_-, \Sigma)$ are the canonical absolute and relative $\mathrm{Spin}^{\mathbb{C}}$ *structure of the Lefschetz fibration. In the last sum, $\mathfrak{s}$ is any fixed element of $\mathfrak{s}_- \#_\Sigma \mathfrak{s}_+ \subset$* $\mathrm{Spin}^{\mathbb{C}}(X)$*.*

3.2. Properties A1, A2, and A5.

A1. The finiteness is a fundamental well-known property of SW invariants, which holds as well for OS invariants, see [OS3], Theorem 3.3.

A2. It is also a well-known property. In fact, a set-theoretic identification of the conjugate $\mathrm{Spin}^{\mathbb{C}}$-bundles S and $\bar{S}$ gives a point-wise correspondence (possibly alternating the orientations) between the solutions spaces to the SW equations associated with S and $\bar{S}$. For the case of OS invariants, see [OS3], Theorem 3.5.

A5. We formulated the simplest classical version of the adjunction inequality. It can be found in a more general form (including the case of $\Sigma^2 < 0$) in [OS2], Theorems 1.1–1.7 for SW invariants, and in [OS3], Theorem 1.4. for OS invariants.

3.3. Lefschetz normalization properties A3(1)–(3).

A3(1). For SW invariants this property is proved by Taubes [T1] (the Main Theorem). The case of OS invariants was considered in [OS4], Theorem 5.1. For this part of A3 the minimality condition is not required.

A3(2). By Taubes' result [T2], Theorem 2, $|c_1(\mathfrak{s}) \circ [\omega]| \leqslant |c_1(\mathfrak{s}_X) \circ [\omega]|$ for any SW basic structure $\mathfrak{s} \in \mathrm{Spin}^{\mathbb{C}}(X)$, with the equality only for $\mathfrak{s} = \mathfrak{s}_X$ and $\mathfrak{s} = \bar{\mathfrak{s}}_X$ (here $\circ$ denotes the pairing in $H^2(X)$). According to [OS4, Theorem 1.1], the same holds for OS basic structures. Gompf's construction produces a symplectic form in a Lefschetz fibration $p : X \to S^2$ as a small perturbation of $\omega = p^*(\omega_{S^2}) + t\eta$, where ω_{S^2} is the area form in S^2, η is a closed 2-form in X having positive restrictions to the fibers of p at every point, and $0 < t << 1$. Observing that $[\Sigma]$ is dual to a properly normalized 2-form $p^*\omega_{S^2}$ and, thus, $c_1(\mathfrak{s}) \circ p^*\omega_{S^2} = c_1(\mathfrak{s})[\Sigma]$, we can deduce letting $t \to 0$ that $|c_1(\mathfrak{s})[\Sigma]| \leqslant |c_1(\mathfrak{s}_X)[\Sigma]|$. Using that $[\eta] \in H^2(X)$ can be any class with $[\eta][\Sigma] > 0$, we can also conclude that the equality $|c_1(\mathfrak{s})[\Sigma]| = |c_1(\mathfrak{s}_X)[\Sigma]|$ may hold only in the case of $\mathfrak{s} = \mathfrak{s}_X + n\sigma$, or $\mathfrak{s} = \bar{\mathfrak{s}}_X + n\sigma$, where $\sigma \in H^2(X)$ is dual to $[\Sigma]$. But the symplectic manifolds are of the simple type [T4], Theorem 02(6), which implies that $\mathfrak{s}_X + n\sigma$ (and similarly $\bar{\mathfrak{s}}_X + n\sigma$) cannot be basic for $n \neq 0$, since $[c_1^2(\mathfrak{s}_X + n\sigma) - c_1^2(\mathfrak{s}_X)] = 2n\chi(\Sigma) \neq 0$ in case of $g(\Sigma) > 1$.

A3(3) for OS invariants. It is proved by the arguments in Lemma 5.7 from [OS4] for OS invariants. The same scheme of the proof works for SW invariants as well, so we will briefly review it (sending a reader to [OS4] for the notation and details).

The first step is to observe that the canonical structure $\mathfrak{s}_X \in \mathrm{Spin}^{\mathbb{C}}(X)$ is the only one satisfying the adjunction inequality with respect to a certain family of surfaces $F \subset X$. For these surfaces $F^2 < 0$, and so in principle the inequality may fail for a basic structure $\mathfrak{s} \in \mathrm{Spin}^{\mathbb{C}}(X)$, but in this case there is another basic $\mathrm{Spin}^{\mathbb{C}}$ structure $\mathfrak{s}' = \mathfrak{s} + f$, where $f \in H^2(X)$ is Poincare dual to $[F]$, and there exists $\xi \in \mathbb{A}_F$ such that $\Phi_{X,\mathfrak{s}'}(\xi x) = \Phi_{X,\mathfrak{s}}(x)$ for any $x \in \mathbb{A}_X$ (the action of ξ on x means the action of the image of ξ under the inclusion map $\mathbb{A}_F \to \mathbb{A}_X$). One can notice next that the construction of surfaces F in [OS4] yields

a natural epimorphism $H_1(\Sigma) \to H_1(F)$ commuting with the inclusion homomorphisms from $H_1(\Sigma)$ and $H_1(F)$ to $H_1(X)$, and so we may assume that $\xi \in \mathbb{A}_\Sigma$.

The second key observation is triviality of the action of $\mathbb{A}_\Sigma$ in $HF^+(M, \mathfrak{t})$, where $M = \Sigma \times S^1$ and $\mathfrak{t} = \mathfrak{s}|_M$ is the canonical structure induced from $\mathfrak{s}_\Sigma \in \mathrm{Spin}^{\mathbb{C}}(\Sigma)$ by the projection $M \to \Sigma$. This triviality is deduced in [OS4] as an immediate corollary of the isomorphism $HF^+(M, \mathfrak{t}) \cong \mathbb{Z}$.

The third ingredient of the proof is the relation between the invariants $F^{\mathrm{mix}}_{X^{\circledcirc}, \mathfrak{s}'^{\circledcirc}}$ and $\Phi_{X,\mathfrak{s}'}$ (the latter is dual to $OS_X(\mathfrak{s}')$ in our notation), where $X^{\circledcirc} = X \smallsetminus (\mathrm{Int}\, N \cup \mathrm{Int}\, B^4)$ is the complement of a tubular neighborhood N of Σ and a ball $B^4 \subset X$ disjoint from N. $X^{\circledcirc}$ is viewed as a cobordism from S^3 to $M = \partial N$, so that $F^{\mathrm{mix}}_{X^{\circledcirc}, \mathfrak{s}'^{\circledcirc}}$ takes values in $HF^+(M, \mathfrak{t}) \cong \mathbb{Z}$. More precisely, $\Phi_{X,\mathfrak{s}'}(\xi x)$ and thus $\Phi_{X,\mathfrak{s}}(x)$, or equivalently, $OS_X(\mathfrak{s})$, vanishes as it is the homogeneous part of $F^{\mathrm{mix}}_{X^{\circledcirc}, \mathfrak{s}'^{\circledcirc}}(\Theta_- \otimes (\xi x)) = \xi F^{\mathrm{mix}}_{X^{\circledcirc}, \mathfrak{s}'^{\circledcirc}}(\Theta_- \otimes x)$, where $\Theta_- \in HF^-(S^3)$ is the generator in the upper dimension, and $\mathfrak{s}'^{\circledcirc} = \mathfrak{s}'|_{X^{\circledcirc}}$ (see the proof of Lemma 5.6 of [OS4]). This contradicts to the assumption that $\mathfrak{s}$ (and thus $\mathfrak{s}'$) is a basic structure.

Applying these arguments to $X \#_\Sigma Y$, we conclude similarly that if $OS_{X \#_\Sigma Y}(\mathfrak{s}) \neq 0$, for $\mathfrak{s} \in \mathrm{Spin}^{\mathbb{C}}(X \#_\Sigma Y)$ such that $\mathfrak{s}|_\Sigma = \mathfrak{s}_\Sigma$, then $\mathfrak{s}|_{X^\circ}$ is canonical, since otherwise $F^{\mathrm{mix}}_{X^\circ \#_\Sigma Y, \mathfrak{s}'^\circ}$ and, thus, $OS_{X \#_\Sigma Y}(\mathfrak{s})$ vanish. $\square$

A3(3) for SW invariants. The first step for SW invariants is like for OS invariants, since the generalized adjunction inequalities look similar in the both theories (cf. [OS1], [OS2] and [OS4]). Next, the Seiberg-Witten-Floer homology groups $HF^{SW}_*(M, \mathfrak{t}) \cong \mathbb{Z}$, (see [MW], Theorem 1.7), and so the action of $\mathbb{A}_\Sigma$ considered in [CW] is trivial on this group for the same reason as in the case of the OS invariants.

The final step goes also like in the OS-theory, but instead of $F^{\mathrm{mix}}_{X^{\circledcirc}, \mathfrak{s}'^{\circledcirc}}(\Theta_- \otimes x)$ we consider the function $\phi^{SW}_{X^\circ}(\mathfrak{s}'^\circ, x^\circ)$ from [CW], where $X^\circ = X \smallsetminus \mathrm{Int}\, N$, $x^\circ = x|_{X^\circ}$ and ${\mathfrak{s}'_X}^\circ = \mathfrak{s}'|_{X^\circ}$. To deduce vanishing of $SW_{X,\mathfrak{s}'}(\xi x)$ we can use the product formula, in Theorem 1.2 of [CW], which implies for $X \#_\Sigma Y = X^\circ \cup Y^\circ$ that

$$SW_{X \#_\Sigma Y, \mathfrak{s}'}(\xi x \otimes y) = \langle [u] \pi_1(\phi^{SW}_{X^\circ}({\mathfrak{s}'_X}^\circ, \xi x)), \pi_2(\phi^{SW}_{Y^\circ}({\mathfrak{s}'_Y}^\circ, y)) \rangle$$

and we can conclude that the product vanishes, because the first factor vanishes. $\square$

3.4. The product formula A4. The well known product formula [MST], Theorem 3.1, concerns the version of SW invariants corresponding to the case of $R_X = \mathbb{Z}[U]$, which is not as general as $R_X = \mathbb{A}^*_X$, although the author supposes that the same arguments without essential changes can be as well used in the most general case. In any case, after [MST], much more general gluing formulae were established, see for instance Theorem 1.2 in [CW], which concerns 4-manifolds with an arbitrary boundary and contains A4 as a corollary.

In the case of OS invariants, A4 can be derived from the product formula [OS3], Theorem 3.4, applied to the fiber sums, although it may look not so obvious as in the case of SW invariants. To clarify it, we give some comments, which are basically extracted from [OS3] and [OS4].

Puncturing a fiber sum, $X = X_- \#_\Sigma X_+ = X_-^\circ \cup X_+^\circ$, at a pair of points, we obtain $X^{\circledcirc} = X_-^{\circledcirc} \cup X_+^{\circledcirc}$, where $X_\pm^{\circledcirc} = X_\pm^\circ \smallsetminus B^4$. $X^{\circledcirc}$ can be viewed as a product of cobordisms $X_-^{\circledcirc} \colon S^3 \to M \cong S^1 \times \Sigma$ and $X_+^{\circledcirc} \colon M \to S^3$. The product formula [OS3], Theorem 3.4,

says

$$[F^+_{X^{\circledcirc}_+,\mathfrak{s}^{\circledcirc}_+}(F^{\rm mix}_{X^{\circledcirc}_-,\mathfrak{s}^{\circledcirc}_-}(\theta_-\otimes x_-)\otimes x_+)]_0=\sum_{\mathfrak{s}|_{X^{\circledcirc}_\pm}=\mathfrak{s}^{\circledcirc}_\pm}\Phi_{X,\mathfrak{s}}(x_-\otimes x_+)$$

where $\mathfrak{s}^{\circledcirc}_\pm\in\mathrm{Spin}^{\mathbb{C}}(X^{\circledcirc}_\pm)$, $\mathfrak{s}^{\circledcirc}_\pm|_M=\mathfrak{t}$, and $[x]_0\in HF^+_0(S^3)$ denotes the 0-dimensional component of $x\in HF^+(S^3)$.

We assume here that $\mathrm{Spin}^{\mathbb{C}}$ structure $\mathfrak{s}^{\circledcirc}_\pm$ is induced in $X^{\circledcirc}_\pm$ from $\mathfrak{s}_\pm\in\mathrm{Spin}^{\mathbb{C}}(X_\pm,\mathfrak{s}_\Sigma)$, and, thus, its restriction, $\mathfrak{t}$ is the canonical $\mathrm{Spin}^{\mathbb{C}}$ structure determined by the SO_2 reduction of τ_M. This implies, in particular, that $HF^+(M,\mathfrak{t})\cong\mathbb{Z}$ [OS4], Lemma 5.5.

The duality between $F^+_{X^{\circledcirc}_+,\mathfrak{s}^{\circledcirc}_+}\colon HF^+(M,\mathfrak{t})\to HF^+(S^3)$ and $F^-_{X^{\circledcirc}_+,\mathfrak{s}^{\circledcirc}_+}\colon HF^-(S^3)\cong HF^-(-S^3)\to HF^-(-M,\mathfrak{t})\cong HF^-(M,\mathfrak{t})$ (see Theorem 3.5 in [OS3]) implies that

$$[F^+_{X^{\circledcirc}_+,\mathfrak{s}^{\circledcirc}_+}(1\otimes x_+)]_0=F^{\rm mix}_{X^{\circledcirc}_+,\mathfrak{s}^{\circledcirc}_+}(\theta_-\otimes x_+),$$

which gives

$$F^{\rm mix}_{X^{\circledcirc}_+,\mathfrak{s}^{\circledcirc}_+}(\theta_-\otimes x_+)F^{\rm mix}_{X^{\circledcirc}_-,\mathfrak{s}^{\circledcirc}_-}(\theta_-\otimes x_-)=\sum_{\mathfrak{s}|_{X^{\circledcirc}_\pm}=\mathfrak{s}^{\circledcirc}_\pm}\Phi_{X,\mathfrak{s}}(x_-\otimes x_+)$$

On the other hand, applying the product formula [OS3], to $X^{\circledcirc}_\pm=X^{\circledcirc}_\pm\cup N^{\circledcirc}_\pm(\Sigma)$, viewed as a product cobordism of $X^{\circledcirc}_\pm\colon S^3\to M$ and $N^{\circledcirc}_\pm\colon M\to S^3$, we obtain

$$F^{\rm mix}_{X^{\circledcirc}_\pm,\mathfrak{s}^{\circledcirc}_\pm}(\theta_-\otimes x_\pm)=\sum_{\mathfrak{s}|_{X^{\circledcirc}_\pm}=\mathfrak{s}^{\circledcirc}}\Phi_{X_\pm,\mathfrak{s}_\pm}(x_\pm)$$

using that the second cobordism induces an isomorphism from $HF^+(M,\mathfrak{t})\cong\mathbb{Z}$ to $HF^+_0(S^3)$ (see [OS4], Theorem 5.3).

The structures $\mathfrak{s}\in\mathrm{Spin}^{\mathbb{C}}(X_\pm)$ in the latter sum differ just by multiples of the class $\sigma\in H^2(X_\pm)$ Poincare-dual to $[\Sigma]$, and, thus, have distinct degrees, $d(\mathfrak{s})$, since $d(\mathfrak{s}+n\sigma)=d(\mathfrak{s})+n\chi(\Sigma)$. In the other words, the latter formula is a decomposition of $F^{\rm mix}_{X^{\circledcirc}_\pm,\mathfrak{s}^{\circledcirc}_\pm}$ into a sum of its homogeneous components (this idea was used in the proof of Lemma 5.6 in [OS4]). Passing from $\Phi_{X,\mathfrak{s}}$ to the dual $OS_X(\mathfrak{s})$ and comparing the components of the same degree, we obtain A4.

§4. Proof of Theorems 1.4.1

4.1. Independence of the choice of W. Consider a pair of Lefschetz fibrations, $W_i\to S^2$, $i=1,2$, with a fiber Σ such that $H_1(W_i)=0$, and denote by $\mathfrak{r}_i\in\mathrm{Spin}^{\mathbb{C}}(W_i,\Sigma)$ the canonical structures. Let $Y_i=X\#_\Sigma W_i$ and $W=W_1\#_\Sigma W_2$, then $Z=X\#_\Sigma W\cong X\#_\Sigma W_1\#_\Sigma W_2\cong Y_1\#_\Sigma W_2$.

4.1.1. Proposition. *For any* $\mathfrak{r}\in\mathrm{Spin}^{\mathbb{C}}(X,\Sigma)$ *we have*

$$S_{Y_1}(\mathfrak{r}\#_\Sigma\mathfrak{r}_1)=S_Z(\mathfrak{r}\#_\Sigma\mathfrak{r}_1\#_\Sigma\mathfrak{r}_2)=S_{Y_2}(\mathfrak{r}\#_\Sigma\mathfrak{r}_2)$$

Proof. Since the two equalities are analogous, we prove only the first one. Let $W_i^\circ=W_i\smallsetminus N_i$, $i=1,2$, denote the complements of an open tubular neighborhood N_i of a fiber

$\Sigma \subset W_i$, and $Y_i^\circ = X\#_\Sigma W_i^\circ$, $W^\circ = W_1^\circ \#_\Sigma W_2$. From Corollary 3.1.1 applied to a fiber sum of $Y_1 \#_\Sigma W_2$ we obtain

$$S_{Y_1}(\mathfrak{r}\#_\Sigma \mathfrak{r}_1) = \sum_{h\in\Delta_M} S_Z((\mathfrak{r}\#_\Sigma\mathfrak{r}_1\#_\Sigma\mathfrak{r}_2)+h).$$

where $M = \partial W_2^\circ$ and $\Delta_M \subset H^2(Z)$ is the image of $H^1(\Sigma)$ under the composition $\delta_M \circ q^*\colon H^1(\Sigma) \to H^1(M) \to H^2(Z)$, like in 2.11.

Note that the sum in the above formula has only one non-vanishing term corresponding to $h = 0$, because the restriction of any basic structure $(\mathfrak{r}\#_\Sigma\mathfrak{r}_1\#_\Sigma\mathfrak{r}_2)+h$ to W should coincide with $\mathfrak{r}_1\#_\Sigma\mathfrak{r}_2|_{W^\circ}$ according to A3(3) and 2.12.1. On the other hand, for $h \neq 0$ it does not coincide, because of the following observation.

4.1.2. Lemma. *The following composition is injective*

$$H^1(\Sigma) \xrightarrow{\ q^*\ } H^1(M) \xrightarrow{\ \delta_M\ } H^2(Z) \longrightarrow H^2(W^\circ)$$

(the last map here is the inclusion homomorphism).

Proof. The Poincare dual to these homomorphisms are the homomorphisms

$$H_1(\Sigma) \to H_2(M) \to H_2(Z) \to H_2(W^\circ, \partial W^\circ)$$

sending $h_1 \in H_1(\Sigma)$ to the image of $h_2 = h_1 \times [S^1] \in H_2(\Sigma \times S^1) \cong H_2(M)$ under the inclusion map $H_2(M) \to H_2(W^\circ)$ composed with the relativization map $H_2(W^\circ) \to H_2(W^\circ, \partial W^\circ)$. The condition that $H_1(W_i) = H_1(W_i^\circ) = 0$ allows to find a cycle in $H_2(W^\circ)$ having non-vanishing intersection index with h_2 in W, if $h_1 \neq 0$, thus proving non-vanishing of the image of $[h_2]$ in $H_2(W^\circ, \partial W^\circ)$. $\square$

4.2. Proof of Properties (1)–(5) in Theorems 1.4.1.

(1) This property is just A1 applied to $X\#_\Sigma W$.

(2) This holds by definition of the invariants in the case of $\Sigma^2 > 0$.

(3) Note that the connected sum $X\#_\Sigma W$ is the same as the sum $X\#_{-\Sigma}\overline{W}$, where $\overline{W}$ is the conjugate to W Lefschetz fibration. Axiom $A2$ implies that $S_{X,\Sigma}(\mathfrak{r}) = S_{X\#_\Sigma W}(\mathfrak{r}\#_\Sigma\mathfrak{r}_{W,\Sigma})$ is equal to $\pm S_{X\#_\Sigma W}(\overline{\mathfrak{r}\#_\Sigma\mathfrak{r}_{W,\Sigma}})$ where the conjugate Spin$^{\mathbb{C}}$ structure $\overline{\mathfrak{r}\#_\Sigma\mathfrak{r}_{W,\Sigma}}$ equals to $\overline{\mathfrak{r}}\#_{-\Sigma}\overline{\mathfrak{r}_{W,\Sigma}}$ as follows from 2.12, and $\overline{\mathfrak{r}_{W,\Sigma}} = \mathfrak{r}_{\overline{W},\overline{\Sigma}}$, as remarked in 2.8. On the other hand, using $\overline{W}$ to evaluate $S_{X,-\Sigma}(\overline{\mathfrak{r}})$, we obtain $S_{X,-\Sigma}(\overline{\mathfrak{r}}) = S_{X\#_{-\Sigma}\overline{W}}(\overline{\mathfrak{r}}\#_{-\Sigma}\mathfrak{r}_{\overline{W},\overline{\Sigma}})$ that is $\pm S_{X,\Sigma}(\mathfrak{r})$.

(4) It follows immediately from Proposition 2.12.1 and A3(1).

(5) It follows from Corollary 3.1.1 applied to the fiber sum $X\#_\Sigma W$, namely

$$S_X(\mathfrak{s}) = \sum_{\mathfrak{r}\in abs^{-1}_{X,\Sigma}(\mathfrak{s})} S_{X\#_\Sigma W}(\mathfrak{r}\#_\Sigma\mathfrak{r}_{W,\Sigma}) = \sum_{\mathfrak{r}\in abs^{-1}_{X,\Sigma}(\mathfrak{s})} S_{X,\Sigma}(\mathfrak{r}),$$

where $\mathfrak{r}_{W,\Sigma} \in \mathrm{Spin}^{\mathbb{C}}(W,\Sigma)$ is the canonical relative Spin$^{\mathbb{C}}$ structure of a Lefschetz fibration and $abs_{X,\Sigma}\colon \mathrm{Spin}^{\mathbb{C}}(X,\Sigma) \to \mathrm{Spin}^{\mathbb{C}}(X)$ the forgetful map. $\square$

4.3. Proof of the product formula (6). Consider a fiber sum $X = X_+ \#_\Sigma X_-$ and Lefschetz fibrations $W_\pm \to S^2$ with a fiber Σ and $H^1(W_\pm) = 0$. Put $Y_\pm = X_\pm \#_\Sigma W_\pm$, $W = W_+ \#_\Sigma W_-$, and $Z = X \#_\Sigma W \cong Y_+ \#_\Sigma Y_-$.

Choose a pair of relative structure $\mathfrak{r}_\pm \in \mathrm{Spin}^{\mathbb{C}}(X_\pm, \Sigma)$, denote by $\mathfrak{r}_{W_\pm,\Sigma} \in \mathrm{Spin}^{\mathbb{C}}(W_\pm, \Sigma)$ the canonical $\mathrm{Spin}^{\mathbb{C}}$ structures of Lefschetz fibrations in $W_\pm$ and let $\mathfrak{s}_\pm = \mathfrak{r}_\pm \#_\Sigma \mathfrak{r}_{W_\pm,\Sigma} \in \mathrm{Spin}^{\mathbb{C}}(Y_\pm)$, $\mathfrak{s} = \mathfrak{r}_+ \#_\Sigma \mathfrak{r}_{W_+,\Sigma} \#_\Sigma \mathfrak{r}_- \#_\Sigma \mathfrak{r}_{W_-,\Sigma} \in \mathrm{Spin}^{\mathbb{C}}(Z)$. By Proposition 2.12.1, $\mathfrak{r}_{W,\Sigma} = \mathfrak{r}_{W_-,\Sigma} \vee \mathfrak{r}_{W_+,\Sigma} \in \mathrm{Spin}^{\mathbb{C}}(W, \Sigma)$ is the canonical relative $\mathrm{Spin}^{\mathbb{C}}$ structure of the Lefschetz fibration in W and $\mathfrak{s} = (\mathfrak{r}_+ \vee \mathfrak{r}_-) \#_\Sigma \mathfrak{r}_{W,\Sigma}$.

The product formula A4 applied to the fiber sum $Z = Y_+ \#_\Sigma Y_-$ reads

$$\sum_{k \in \mathbb{Z}} S_{Y_+}(\mathfrak{s}_+ + k\sigma_+) S_{Y_-}(\mathfrak{s}_- - k\sigma_-) = \sum_{\mathfrak{s}' \in \mathfrak{s}_+ \# \mathfrak{s}_-} S_Z(\mathfrak{s}')$$

The sum in the right-hand side contains only one term $S_Z(\mathfrak{s})$ which follows from the arguments analogous to those in 4.1. Finally, we observe that $S_{X_\pm,\Sigma}(\mathfrak{r}_\pm \pm k\sigma_\pm) = S_{Y_\pm}(\mathfrak{s}_\pm \pm k\sigma_\pm)$, $S_{X,\Sigma}(\mathfrak{r}_+ \vee \mathfrak{r}_-) = S_Z(\mathfrak{s})$.

Equivalence of the alternative formulations of the product formula in 1.4.1(6) follows from that $\mathfrak{r}'_+ \vee \mathfrak{r}'_- = \mathfrak{r}_+ \vee \mathfrak{r}_-$ if and only if $\mathfrak{r}'_\pm = \mathfrak{r}_\pm \pm k\sigma_\pm$ for some $k \in \mathbb{Z}$. $\square$

4.4. Proof of the Adjunction inequality (7). The idea of the proof is to find an appropriate Lefschetz fibration $W \to S^2$ with a fiber $\Sigma_W \cong \Sigma$ having a membrane, $F_W \subset W$ whose boundary, ∂F_W matches with the boundary ∂F of membrane $F \subset X$. Then after gluing F and F_W we can get a closed surface $\hat{F} \subset X \#_\Sigma W$, which will be oriented if the orientations of ∂F and ∂F_W match.

More precisely, we should glue the complements X° and W° of tubular neighborhoods $N \subset X$ of Σ and $N_W \subset W$ of Σ_W so that $F \cap \partial N$ is glued to $F_W \cap \partial N_W$. It is not difficult to see that connectedness of $\Sigma \smallsetminus \partial F$ guarantees that we can find such a gluing map $\partial N \to \partial N_W$.

Finally, we want to make use of the adjunction inequality A5 for $\hat{F}$. This requires $\hat{F}^2 = F^2 + F_W^2 \geqslant 0$, which holds if we can find F_W with a sufficiently big self-intersection index. If we choose F_W so that $\mathfrak{r}_{W,\Sigma}[F_W] = 0$ for the canonical structure $\mathfrak{r}_{W,\Sigma}$ of the Lefschetz fibration, then $c_1(\mathfrak{r} \# \mathfrak{r}_{W,\Sigma})[\hat{F}] = \mathfrak{r}[F] + \mathfrak{r}_{W,\Sigma}[F_W] = \mathfrak{r}[F]$ and thus

$$-\chi(\hat{F}) = -\chi(F) - \chi(F_W) \geqslant F^2 + F_W^2 + |\mathfrak{r}[F]|$$

which gives (7) of Theorem 1.4.1 provided $F_W^2 = -\chi(F_W)$. So, we reduced the problem to constructing the following example of W and F_W.

4.4.1. Proposition. *Let Σ be a surface of genus $g \geqslant 1$, $L \subset \Sigma$ an oriented curve (possibly multi-component) with the connected complement $\Sigma \smallsetminus L$, and $n \in \mathbb{N}$. Then there exists a relatively minimal Lefschetz fibration $p\colon W \to \mathbb{CP}^1$, with $H_1(W) = 0$, whose fiber, Σ, has a membrane, $F_W \subset W$, such that*

1. $\partial F_W = L$ *(as an oriented curve),*
2. $F_W^2 = -\chi(F_W)$,
3. $\mathfrak{r}_{W,\Sigma}[F_W] = 0$, *where $\mathfrak{r}_{W,\Sigma} \in \mathrm{Spin}^{\mathbb{C}}(W, \Sigma)$ is the canonical* $\mathrm{Spin}^{\mathbb{C}}$ *structure,*
4. $-\chi(F_W) > n$.

4.5. Real Lefschetz fibrations. We will construct a complex algebraic Lefschetz fibration $p\colon W \to \mathbb{C}\mathrm{P}^2$ endowed with a *real structure*, that is just an anti-holomorphic involution (*the complex conjugation in* W), $c\colon W \to W$, which commutes with p and the complex conjugation in $\mathbb{C}\mathrm{P}^1$. The *real locus*, $\mathbb{R}W$, of W is the fixed point set of c. For a real fiber, $\Sigma = p^{-1}(b)$, $b \in \mathbb{R}\mathrm{P}^1$, we let $\mathbb{R}\Sigma = \Sigma \cap \mathbb{R}W$. A membrane F_W in our example will be the closure of a properly chosen connected component of $\mathbb{R}W \smallsetminus \mathbb{R}\Sigma$ bounded by $\mathbb{R}\Sigma$. Such a choice guarantees the condition (2) of the Proposition 4.4.1, since the tangent bundle to $\mathbb{R}W$ is anti-isomorphic to the normal bundle via the operator $J\colon \tau_X \to \tau_X$ of the complex structure. The condition (3) follows from that the real determinant gives a section trivializing the complex determinant bundle (and thus the associated $\mathrm{Spin}^{\mathbb{C}}$ determinant), provided $F_W \subset \mathbb{R}W$ is orientable.

Note furthermore that the pairs (Σ, L) are classified up to homeomorphism respecting the orientations of Σ and L just by the genus g and the number of components, $r \leqslant g$, of L, under our assumption that $\Sigma \smallsetminus L$ is connected. So, we will achieve $(\Sigma, \mathbb{R}\Sigma) \cong (\Sigma, L)$, that is the condition (1), if $\mathbb{R}\Sigma$ does not divide Σ into halves and has the required number r of the components. This reduces Proposition 4.4.1 to the following construction.

4.5.1. Proposition. *For any integers $k \in \mathbb{N}$ and $g \geqslant r \geqslant 1$, there exists a relatively minimal real algebraic Lefschetz fibration, $p\colon W \to \mathbb{C}\mathrm{P}^1$, $\mathrm{conj}_W\colon W \to W$, with $H_1(W) = 0$, and a real fiber $\Sigma = p^{-1}(b)$ of genus g such that*

1. *$\mathbb{R}\Sigma$ has r components, and $\Sigma \smallsetminus \mathbb{R}\Sigma$ is connected,*
2. *there is a connected orientable component of $\mathbb{R}W \smallsetminus \mathbb{R}\Sigma$, whose closure, F_W, is bounded by $\mathbb{R}\Sigma$,*
3. *$g(F_W) \geqslant k$.*

4.6. Proof of Proposition 4.5.1. Consider a double covering $q\colon W \to \mathbb{C}\mathrm{P}^1 \times \mathbb{C}\mathrm{P}^1$, branched along a non-singular curve $\mathbb{C}A$ defined over $\mathbb{R}$ and having degree $(2g+2, 2d)$, where d is sufficiently large. The Lefschetz fibration that we need is the composition of q with the projection to the first factor. A generic fiber, $\Sigma_t = q^{-1}(t \times \mathbb{C}\mathrm{P}^1)$, $t \in \mathbb{C}\mathrm{P}^1$, projects to $\mathbb{C}\mathrm{P}^1$ as a double cover branched at $(2g+2)$ points, $\mathbb{C}A_t = \mathbb{C}A \cap (t \times \mathbb{C}\mathrm{P}^1)$, and thus has genus g. If this branching locus has $2r$ real points, $\mathbb{R}A_t = \mathbb{C}A_t \cap \mathbb{R}\mathrm{P}^1$, then Σ_t satisfies the condition (1) of Proposition 4.5.1.

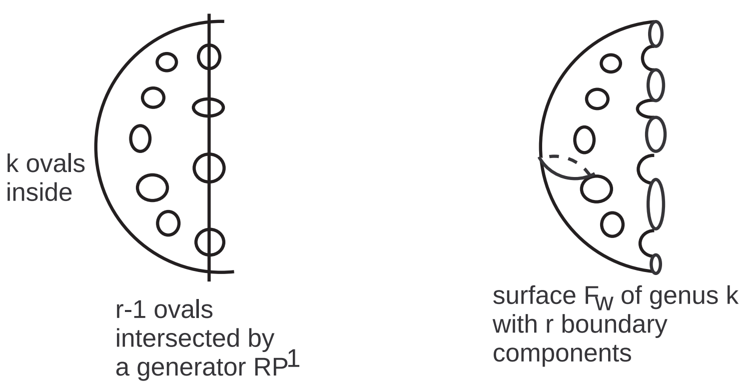

FIGURE 1

The conditions (2)–(3) will be obviously satisfied if we choose the curve $\mathbb{R}A$ and a

generator $t \times \mathbb{R}\mathrm{P}^1$ with the mutual position shown on the Figure 1. The curve with such position can be easily constructed by a perturbation of a nodal curve in $\mathbb{C}\mathrm{P}^1 \times \mathbb{C}\mathrm{P}^1$ splitting into a union of generators. $\square$

§5. Some generalizations

5.1. Variants of $S_{X,\Sigma}$. One can consider a version of the invariant $S_{X,\Sigma}$,

$$S_{X,\Sigma,K}\colon \operatorname{Spin}^{\mathbb{C}}(X,\Sigma)/K \to R_{X,\Sigma,K}$$

which depends on a subgroup $K \subset H^1(\Sigma)$. Here $\operatorname{Spin}^{\mathbb{C}}(X,\Sigma)/K$ is the quotient of $\operatorname{Spin}^{\mathbb{C}}(X,\Sigma)$ by the action of K, where $h \in K$ acts as $\mathfrak{s} \mapsto \mathfrak{s} + \delta(h)$, and $\delta\colon H^1(\Sigma) \to H^2(X,\Sigma)$ is the boundary homomorphism. The ring $R_{X,\Sigma,K}$ is $\mathbb{A}^*_{X,\Sigma,K} = \Lambda(H^1(X,\Sigma)/K) \otimes \mathbb{Z}[U]$ in the case of $S_X = SW_X$ and $\mathbb{A}^*_{X,\Sigma,K} \otimes \mathbb{Z}/2$ in the case of $S_X = OS_X$. Considering the invariant in the reduced form, we let $R_{X,\Sigma,K}$ be just $\mathbb{Z}$ for SW and $\mathbb{Z}/2$ for OS invariants.

The definition of $S_{X,\Sigma,K}$ is similar to that of $S_{X,\Sigma}$, except that the condition $H_1(W) = 0$ for a Lefschetz fibration $W \to S^2$ is replaced by the condition $K = \operatorname{Im}(H^1(W) \to H^1(\Sigma))$ (if K can be expressed as such an image). In particular, for $K = 0$, we have $S_{X,\Sigma,K} = S_{X,\Sigma}$ and for $K = H^1(\Sigma)$ the invariant $S_{X,\Sigma,K}$ coincides with the restriction, $S_{X,\mathfrak{s}_\Sigma}$, of the absolute invariant.

In general, there is a splitting formula $S_{X,\Sigma,K}[\mathfrak{s}] = \sum_{\mathfrak{s}' \in [\mathfrak{s}]} S_{X,\Sigma}(\mathfrak{s}')$ or equivalently, $\underline{S}_{X,\Sigma,K} = (abs_K)_*(\underline{S}_{X,\Sigma})$, where $(abs_K)_*$ is the push-forward morphism of the projection $abs_K\colon \operatorname{Spin}^{\mathbb{C}}(X,\Sigma) \to \operatorname{Spin}^{\mathbb{C}}(X,\Sigma)/K$. The proof of this formula is analogous to the proof of (5) in Theorem 1.4.1.

5.2. A refinement of the Ozsváth-Szabó 4-dimensional invariant with respect to a mapping torus boundary component. The idea used in the definition of $S_{X,\Sigma}$ can be used also to define the refinement of the 4-dimensional Ozsváth-Szabó invariants in a more general setting. Assume for instance that $X\colon M_0 \to M_1$ is a cobordism between 3-manifolds, where $M_1 = M_f$ is a mapping torus of some homeomorphism $f\colon \Sigma \to \Sigma$.

The plane field tangent to the fibers of the projection $M_f \to S^1$ defines a canonical $\operatorname{Spin}^{\mathbb{C}}$ extension of the tangent bundle τ_M. Let $\operatorname{Spin}^{\mathbb{C}}(X, M_f)$ denote the set of the relative $\operatorname{Spin}^{\mathbb{C}}$ structures in X with respect to such $\operatorname{Spin}^{\mathbb{C}}$ extension over $M_f \subset \partial X$.

Choose any $\mathfrak{r} \in \operatorname{Spin}^{\mathbb{C}}(X, M_1)$ and let $\mathfrak{s} = abs(\mathfrak{r}) \in \operatorname{Spin}^{\mathbb{C}}(X)$, and $\mathfrak{t}_i = \mathfrak{s}|_{M_i}$, $i = 0, 1$ (here $\mathfrak{t}_1$ is the canonical structure on M_f). Consider an auxiliary cobordism $W\colon M_1 \to M_2$ which has structure of a Lefschetz fibration $q\colon W \to S^1 \times [1,2]$ over the annulus, so that $M_i = q^{-1}(S^1 \times i)$. There is a canonical relative $\operatorname{Spin}^{\mathbb{C}}$ structure, $\mathfrak{r}_W \in \operatorname{Spin}^{\mathbb{C}}(W, M_1)$, which is a refinement of the canonical absolute structure $\mathfrak{s}_W = abs(\mathfrak{r}_W) \in \operatorname{Spin}^{\mathbb{C}}(W)$. We assume that the Lefschetz fibration is relatively minimal and $b_1(W) = 0$ (one can always find such a fibration bounded by any prescribed mapping tori M_i, $i = 1, 2$; for example we may assume in addition that $M_2 \cong \Sigma \times S^1$).

The homomorphism

$$F^+_{X,\mathfrak{r}}\colon HF^+(M_0, \mathfrak{t}_0) \to HF^+(M_1, \mathfrak{t}_1) \cong \mathbb{Z}$$

is defined as the composition of $F^+_{X \cup W, \mathfrak{r}\mathfrak{r}_W}\colon HF^+(M_0, \mathfrak{t}_0) \to HF^+(M_2, \mathfrak{t}_2)$ and the inverse $(F^+_{W,\mathfrak{s}_W})^{-1}$ to the isomorphism $F^+_{W,\mathfrak{s}_W}\colon HF^+(M_1, \mathfrak{t}_1) \to HF^+(M_2, \mathfrak{t}_2)$.

The action of $H_1(X \cup W) = H_1(X \cup W, W) = H_1(X, M_1)$ in $F^+_{X \cup W, \mathfrak{r}\mathfrak{r}_W}$ composed with the isomorphism $(F^+_{W,\mathfrak{s}_W})^{-1}$ defines an action of $\mathbb{A}_{X,M_1}$ in $F_{X,\mathfrak{r}}$.

The product formula of [OS3] applied to the cobordism $X \cup W\colon M_0 \to M_2$ implies a splitting $F^+_{X,\mathfrak{s}} = \sum_{\mathfrak{r} \in abs^{-1}(\mathfrak{s})} F^+_{X,\mathfrak{r}}$. Similarly one can define maps $F^-_{X,\mathfrak{r}}$ and $F^{\text{mix}}_{X,\mathfrak{r}}$ and obtain analogous decompositions of $F^-_{X,\mathfrak{s}}$ and $F^{\text{mix}}_{X,\mathfrak{s}}$.

All the constructions in this section admit also similar versions for SW invariants.

References

[CW] A. L. Carey, B.-L. Wang, *Seiberg-Witten-Floer Homology and Gluing Formulae*, Acta Math. Sin. **19** (2003), no. 2, 245–296.

[FS1] R. Fintushel, R. Stern, *Knots, Links, and 4-manifolds*, Invent. Math. **134** (1998), no. 2, 363–400.

[FS2] R. Fintushel, R. Stern, *Surfaces in 4-manifolds*, Math. Res. Lett. **4** (1997), 907–914.

[G] R. Gompf, *Sums of elliptic surfaces*, J. Diff. Geom. **34** (1991), 93–114.

[KM] P.B. Kronheimer, T.S. Mrowka, *The genus of embedded surfaces in the projective plane*, Math. Research Letters **1** (1994), 797–808.

[MST] J.W. Morgan, Z.Szabó, C.H. Taubes, *A product formula for Seiberg-Witten invariants and the generalized Thom Conjecture*, J. Diff. Geom. **44** (1996), 706–788.

[MW] V. Munoz, B.-L. Wang, Seiberg-Witten-Floer homology of a surface times a circle for non-torsion $\mathrm{Spin}^{\mathbb{C}}$ structures, ArXive: math.DG/9905050.

[OS1] P. Ozsváth, Z. Szabó, *The symplectic Thom Conjecture*, Ann. of Math. **151** (2000), no. 1, 93–124.

[OS2] P. Ozsváth, Z. Szabó, *Higher type adjunction inequalities in Seiberg-Witten theory*, J. Differential Geom. **55** (2000), no. 3, 385–440.

[OS3] P. Ozsváth, Z. Szabó, *Holomorphic Triangles and invariants for smooth four-manifolds*, math.SG/0110169 (2001).

[OS4] P. Ozsváth, Z. Szabó, *Holomorphic triangles invariant and the topology of symplectic four-manifolds*, Duke Math. J. **121** (2004), 1–34.

[T1] C. Taubes, *The Seiberg-Witten invariants and symplectic forms*, Math. Research. Letters **1** (1994), 809–822.

[T2] C. Taubes, *More constraints on symplectic forms from Seiberg-Witten invariants*, Math. Research. Letters **2** (1995), 9–13.

[T3] C. Taubes, *GR=SW: Counting curves and connections*, J. Diff. Geom. **52** (1999), 453–609.

[T4] C. Taubes, SW $\Rightarrow$ *Gr: from the Seiberg-Witten equations to pseudo-holomorphic curves*, J. Amer. Math. Soc. **9**, no. 2, 845–918.

[T5] C. Taubes, *The Seiberg-Witten invariants and 4-manifolds with essential tori*, Geometry and Topology **5** (2001), 441–519.